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Instanton Numbers and Exchange Symmetries in $N = 2$ Dual String Pairs

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ABSTRACT

In this note, we comment on Calabi-Yau spaces with Hodge numbers $h_{1,1} = 3$ and $h_{2,1} = 243$. We focus on the Calabi-Yau space $WP_{1,1,2,8,12}(24)$ and show how some of its instanton numbers are related to coefficients of certain modular forms. We also comment on the relation of four dimensional exchange symmetries in certain $N = 2$ dual models to six dimensional heterotic/heterotic string duality.

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1. Introduction

Recently, there has been an enormous progress in the understanding of non-perturbative effects in supersymmetric field theories and in superstring theories. In particular, various types of strong-weak coupling duality symmetries are by now quite well established, such as S -duality of the four-dimensional $N = 4$ heterotic string [1, 2, 3], string/string dualities [4, 5, 6] between the heterotic and type II strings and heterotic/heterotic duality in $D = 6$ [7, 8]. It now seems that most or even all non-perturbative duality symmetries originate from underlying theories in higher dimensions (from M -theory [5, 9] in $D = 11$ or from F -theory [10, 11] in $D = 12$).

In the following we will be concerned with $N = 2, D = 4$ string theories which have both a heterotic and a type II description [6, 12]. In this context, there is a particularly interesting class of models, which exhibits a non-perturbative symmetry which exchanges the heterotic dilaton S with one of the vector moduli fields. In the context of heterotic/type II string/string duality this type of exchange symmetry was first discussed in [12], where this exchange symmetry was related to a remarkable symmetry property of rational instanton numbers. Subsequent work connected this symmetry to the monodromy group of the CY compactification [13, 14] and discussed [15] the action of this exchange symmetry on BPS spectra and higher derivative gravitational couplings. In this paper, we will extend the previous work of [6, 12, 13, 16, 17, 18] in several directions. We will focus on a class of type II theories based on elliptically fibered CY spaces. We will, in particular, discuss the CY space described by a hypersurface of degree 24 in weighted projective space $WP_{1,1,2,8,12}(24)$ with, in heterotic language, three vector moduli S, T, U . We will investigate the rational (genus 0) as well as the elliptic (genus 1) instanton numbers for this class of models. We will show that the genus 0 as well as the genus 1 instanton numbers can, in the heterotic weak coupling limit, be precisely expressed by the coefficients of the q expansion of certain modular forms. This means that these instanton numbers are nothing else than the multiplicities of positive roots of some generalized Kac-Moody algebra recently discussed in [19]. (For the rational instanton numbers this relation was already anticipated in [19].) Hence one can expect that the non-perturbative $S - T$ exchange symmetries are reflected in a nice symmetry structure of some non-perturbative infinite-dimensional algebra. In order to relate these instanton numbers to the expansion coefficients of certain modular forms, we will have to work out the precise identification of the heterotic vector moduli with the corresponding type II Kähler class fields. We will also discuss the action of the $S - T$ exchange symmetry in this context. At the end, we will comment on the relation of the four-dimensional exchange symmetry to the

six-dimensional heterotic/heterotic duality symmetry [20, 7, 8] in this class of models.

2. Instanton numbers and exchange symmetries

The higher derivative couplings of vector multiplets X to the Weyl multiplet \mathcal{W} of conformal $N = 2$ supergravity can be expressed as a power series [21, 22]

$$F(X, \mathcal{W}^2) = \sum_{g=0}^{\infty} F_g(X) (\mathcal{W}^2)^g. \quad (1)$$

In the context of the type II string, F_g^{II} only receives perturbative contributions at genus g . In the heterotic context, F_g^{het} is perturbatively determined at the tree and at the one loop level; in addition, F_g^{het} also receives non-perturbative corrections. For models with heterotic/type II duality one expects that $F_g^{II}(t_i) = \alpha_g F_g^{het}(S, T_m)$, where α_g denotes a normalisation constant. The t_i ($i = 1, \dots, h$) denote the Kähler class moduli on the type II side, whereas S and T_m ($m = 1, \dots, h-1$) denote the dilaton and the vector moduli on the heterotic side.

First consider the prepotential F_0 which determines the gauge couplings. The two prepotentials F_0^{II} and F_0^{het} should match up upon a suitable identification of the t_i with S and T_m . On the type II side, the Yukawa couplings are given by [23]

$$F_{klm}^{II} = \mathcal{F}_{klm}^0 + \sum_{d_1, \dots, d_h} \frac{n_{d_1, \dots, d_h}^r d_k d_l d_m}{1 - \prod_i q_i^{d_i}} \prod_{i=1}^h q_i^{d_i}, \quad (2)$$

where $q_i = e^{-2\pi t_i}$. The \mathcal{F}_{klm}^0 denote the intersection numbers, whereas the n_{d_1, \dots, d_h}^r denote the rational instanton numbers of genus zero. These instanton numbers are expected to be integer numbers. We will, in the following, work inside the Kähler cone $\sigma(K) = \{\sum_i t_i J_i | t_i > 0\}$. For points inside the Kähler cone $\sigma(K)$, one has for the degrees d_i that $d_i \geq 0$.

Integrating back yields that

$$F_0^{II} = \mathcal{F}^0 - \frac{1}{(2\pi)^3} \sum_{d_1, \dots, d_h} n_{d_1, \dots, d_h}^r Li_3 \left(\prod_{i=1}^h q_i^{d_i} \right) \quad (3)$$

up to a quadratic polynomial in the t_i . \mathcal{F}^0 is cubic in the t_i . Here one has used that $\partial_{t_k} \partial_{t_l} Li_3 = (-2\pi)^2 d_k d_l Li_1$, where $Li_1(x) = -\log(1-x)$.

In the following we will be focusing on a specific type IIA model, namely the S-T-U model [6] based on the Calabi-Yau space $WP_{1,1,2,8,12}(24)$ with $h = h_{11} = 3$, $h_{21} = 243$ and, hence, with $\chi = -480$. Thus we have three Kähler moduli t_1, t_2, t_3 and instanton

numbers n_{d_1, d_2, d_3}^r . The classical Yukawa couplings \mathcal{F}_{klm}^0 on the type II side are given by [23]

$$\begin{aligned}\mathcal{F}_{t_1 t_1 t_1}^0 &= 8 \ , \ \mathcal{F}_{t_1 t_1 t_2}^0 = 2 \ , \ \mathcal{F}_{t_1 t_1 t_3}^0 = 4 \ , \\ \mathcal{F}_{t_1 t_2 t_3}^0 &= 1 \ , \ \mathcal{F}_{t_1 t_3 t_3}^0 = 2.\end{aligned}\tag{4}$$

It follows that

$$\mathcal{F}^0 = \frac{4}{3}t_1^3 + t_1^2 t_2 + 2t_1^2 t_3 + t_1 t_2 t_3 + t_1 t_3^2.\tag{5}$$

Some of the instanton numbers n_{d_1, d_2, d_3}^r can be found in [23]. When investigating the prepotential F_0^{II} [12], two symmetries become manifest, namely

$$t_1 \rightarrow t_1 + t_3, \ t_3 \rightarrow -t_3 \quad \text{for } t_2 = \infty,\tag{6}$$

and

$$t_2 \rightarrow -t_2, \ t_3 \rightarrow t_2 + t_3.\tag{7}$$

These symmetries are true symmetries of F_0^{II} , since the world-sheet instanton numbers n^r enjoy the remarkable properties [12]

$$n_{d_1, 0, d_3}^r = n_{d_1, 0, d_1 - d_3}^r \quad \text{and} \quad n_{d_1, d_2, d_3}^r = n_{d_1, d_3 - d_2, d_3}^r.\tag{8}$$

Observe that \mathcal{F}^0 is completely invariant under the symmetry (7).

Next, consider the heterotic prepotential F_0^{het} . $N = 2$, $D = 4$ heterotic strings can be constructed by compactifying the ten-dimensional heterotic string on $T_2 \times K_3$. A generic compactification of the $E_8 \times E_8$ heterotic string on K_3 , with equal $SU(2)$ instanton number in both E_8 factors, gives rise to $D = 6$ model with gauge group $E_7 \times E_7$. For general vev's of the massless hyper multiplets this gauge group is completely broken, and one is left with 244 hyper multiplets and no massless vector multiplets. Upon a T_2 compactification down to four dimensions, one gets a model with 244 hypermultiplets and with three vector multiplets S , T and U , where S denotes the heterotic dilaton and T, U denote the moduli of T_2 . This model is the heterotic dual of the type IIA model considered above. The heterotic prepotential has the following structure

$$F_0^{het} = -STU + h^{(1)}(T, U) + \mathcal{F}^{\text{non-pert.}}(e^{-S}, T, U).\tag{9}$$

In the following we will consider the semiclassical limit $S \rightarrow \infty$, i.e. $\mathcal{F}^{\text{non-pert}} = 0$, and we will concentrate on the one-loop corrected prepotential. The heterotic semiclassical prepotential [24, 25, 27] has nontrivial monodromy properties under the perturbative

target space duality symmetries $SL(2, \mathbf{Z})_T \times SL(2, \mathbf{Z})_U \times \mathbf{Z}_2^{T \leftrightarrow U}$. The singularities of the semiclassical prepotential at the lines/points $T = U$, $T = U = 1$ or $T = U = e^{i\pi/6}$ reflect the perturbative gauge symmetry enhancement of $U(1)^2$ to $SU(2) \times U(1)$, $SU(2)^2$ or $SU(3)$ respectively. Derivatives of the semiclassical prepotential can be nicely expressed in terms of automorphic functions of T and U . The semiclassical prepotential can be written in the following explicit form [19]

$$F_0^{het} = -STU + \frac{1}{384\pi^2} \tilde{d}_{ABC}^{2,2} y^A y^B y^C - \frac{1}{(2\pi)^4} \sum_{k,l \geq 0} c_1(kl) Li_3(e^{-2\pi(kT+lU)}) - \frac{1}{(2\pi)^4} Li_3(e^{-2\pi(T-U)}), \quad (10)$$

where $y = (T, U)$ and where the constants $c_1(n)$ are related to the positive roots of a generalized Kac-Moody algebra. These constants are determined by

$$\begin{aligned} \frac{E_4 E_6}{\eta^{24}} = \sum_{n \geq -1} c_1(n) q^n &= \frac{1}{q} - 240 - 141444q - 8529280q^2 - 238758390q^3 \\ &- 4303488384q^4 + \dots \end{aligned} \quad (11)$$

The function F_0^{het} has a branch locus at $T = U$. F_0^{het} given in (10) is defined in the fundamental Weyl chamber $T > U$.² The cubic coefficients $\tilde{d}_{ABC}^{2,2}$ will be determined below. We have ignored a possible constant term as well as a possible additional quadratic polynomial in T and U . The cubic terms cannot be uniquely fixed, since the prepotential contains an ambiguity [24, 25] which is a quadratic polynomial in the period vector $(1, T, U, TU)$. Hence, the ambiguity is at most quartic in the moduli and at most quadratic in T and in U . It follows that the third derivative in T or in U is unique; $\frac{\partial^2 h^{(1)}}{\partial T \partial U}$, however, is still ambiguous. Specifically, in the chamber $T > U$, the cubic terms have the following general form [19]

$$\tilde{d}_{ABC}^{2,2} y^A y^B y^C = -32\pi \left(3(1 + \beta) T^2 U + 3\alpha T U^2 + U^3 \right). \quad (12)$$

The cubic term in U is unique, whereas the parameters α and β correspond to the change induced by adding a quadratic polynomial in $(1, T, U, TU)$. As discussed in [24], it is convenient to introduce a dilaton field S_{inv} , which is invariant under the perturbative T -duality transformations at the one-loop level. It is defined as follows

$$\begin{aligned} S_{inv} &= S - \frac{1}{2} \frac{\partial h^{(1)}(T, U)}{\partial T \partial U} - \frac{1}{8\pi^2} \log(j(T) - j(U)) \\ &= S + \frac{1}{4\pi} (1 + \beta) T + \frac{\alpha}{4\pi} U + \frac{1}{8\pi^2} \sum_{k,l \geq 0} kl c_1(kl) Li_1(e^{-2\pi(kT+lU)}) \\ &\quad - \frac{1}{8\pi^2} Li_1(e^{-2\pi(T-U)}) - \frac{1}{8\pi^2} \log(j(T) - j(U)). \end{aligned} \quad (13)$$

²It is meant here that the real part of T is larger than the real part of U .

In the decompactification limit to $D = 5$ [26], obtained by sending $T, U \rightarrow \infty$ ($T > U$), the invariant dilaton S_{inv} has a particularly simple dependence on T and U . Namely, by using $\log j(T) \rightarrow 2\pi T$, one obtains that

$$S_{inv} \rightarrow S_{inv}^\infty = S + \frac{\beta}{4\pi}T + \frac{\alpha}{4\pi}U. \quad (14)$$

Substituting S_{inv}^∞ back into the heterotic prepotential (10) yields that

$$\begin{aligned} F_0^{het} &= -S_{inv}^\infty TU - \frac{1}{12\pi}U^3 - \frac{1}{4\pi}T^2U - \frac{1}{(2\pi)^4} \sum_{k,l \geq 0} c_1(kl) Li_3(e^{-2\pi(kT+lU)}) \\ &\quad - \frac{1}{(2\pi)^4} Li_3(e^{-2\pi(T-U)}). \end{aligned} \quad (15)$$

Note that the ambiguity in α and β is hidden away in S_{inv}^∞ .

Let us now compare the heterotic and the type II prepotentials and identify the t_i ($i = 1, 2, 3$) with S , T and U . In the following, we will actually match $-4\pi F_0^{het}$ with F_0^{II} . First compare the cubic terms in (3) and (10). By assuming that the t_i and S , T and U are linearly related, the following identification between the Kähler class moduli and the heterotic moduli is enforced by the cubic terms

$$\begin{aligned} t_1 &= U \\ t_3 &= T - U \\ t_2 &= 4\pi S_{inv}^\infty = \check{S} + \beta T + \alpha U \end{aligned} \quad (16)$$

where $\check{S} = 4\pi S$. Recall that we are working inside the Kähler cone $\sigma(K) = \{\sum_i t_i J_i | t_i > 0\}$. Now, in the heterotic weak coupling limit one has that indeed $t_2 > 0$. Demanding $t_3 > 0$ implies that one is choosing the chamber $T > U$ on the heterotic side. The identification of t_1 and t_3 agrees, of course, with the one of [12]. The identification of $4\pi S_{inv}^\infty$ with the Kähler variable t_2 becomes very natural when performing the map to the mirror Calabi-Yau compactification with complex structure coordinates x, y, z . Here, since y is invariant under the CY monodromy group, y should be identified [12] with $e^{-8\pi^2 S_{inv}}$. Thus, equation (13) provides the explicit mirror map; for large T, U the Kähler variable $q_2 = e^{-2\pi t_2}$ and the complex structure field y completely agree.

Next, consider the exponential terms in the prepotential F_0 . In the heterotic weak coupling limit $S \rightarrow \infty$, one has that $t_2 \rightarrow \infty$ and, hence, $q_2 = e^{-2\pi t_2} \rightarrow 0$. Then, (3) becomes

$$F_0^{II} = \mathcal{F}^0 - \frac{1}{(2\pi)^3} \sum_{d_1, d_3} n_{d_1, 0, d_3}^r Li_3(q_1^{d_1} q_3^{d_3}). \quad (17)$$

Some of the instanton coefficients contained in (17) are as follows³ [23]

$$\begin{aligned}
n_{d_1,0,0}^r &= n_{d_1,0,d_1}^r = 480 = -2(-240) \ , \\
n_{0,0,1}^r &= -2 \ , \ n_{0,0,d_3}^r = 0 \ , \ d_3 = 2, \dots, 10 \ ; \\
n_{2,0,1}^r &= 282888 = -2(-141444) \ , \\
n_{3,0,1}^r &= n_{3,0,2}^r = 17058560 = -2(-8529280) \ , \\
n_{4,0,1}^r &= 477516780 = -2(-238758390) \ .
\end{aligned} \tag{18}$$

Note that the fact that $n_{d_1,0,0}^r = n_{d_1,0,d_1}^r$ is a reflection of the $T \leftrightarrow U$ exchange symmetry. Now rewriting $kT + lU = (l+k)U + k(T-U) = (l+k)t_1 + kt_3$ and matching $F_0^{II} = -4\pi F_0^{het}$ yields the following identifications

$$\begin{aligned}
d_1 &= k + l \ , \ d_3 = k \\
n_{d_1,0,d_3}^r &= n_{k+l,0,k}^r = -2c_1(kl).
\end{aligned} \tag{19}$$

Note that $d_3 = k \geq 0$ for points inside the Kähler cone. Also, if $d_3 = k = 0$, then $d_1 = l > 0$. On the other hand, if $d_3 = k > 0$, then $d_1 \geq 0$, that is $l \geq -k$.

Comparison of the instanton coefficients listed above with the c_1 -coefficients occurring in the q -expansion of $F(q) = \frac{E_4 E_6}{\eta^{24}}$ in equation (11), shows that the relation (19) is indeed satisfied.

Let us now determine the action of the symmetries (6) and (7) on the heterotic variables. Clearly the perturbative symmetry (6) corresponds to the exchange $T \leftrightarrow U$ for $S \rightarrow \infty$. The non-perturbative symmetry (7) corresponds to

$$\begin{aligned}
S &\rightarrow -(1 + \beta)S - \frac{\alpha(2 + \beta)}{4\pi}U - \frac{\beta(2 + \beta)}{4\pi}T \\
T &\rightarrow 4\pi S + (1 + \beta)T + \alpha U \\
U &\rightarrow U.
\end{aligned} \tag{20}$$

There is one very convenient choice for the parameters α and β , in which the non-perturbative symmetry (20) takes a very simple suggestive form. Namely, for $\alpha = 0$ and $\beta = -1$, this transformation becomes

$$4\pi S \leftrightarrow T, \tag{21}$$

that is, it just describes the exchange of the heterotic dilaton S with the Kähler modulus T of the two-dimensional torus. This choice for α and β is very reasonable, since it is only

³We are grateful to A. Klemm for providing us with a list of instanton numbers for this model.

in this case that the real parts of S and T remain positive after the exchange (20). At the end of this paper, by considering [8] some six-dimensional one-loop gauge couplings, we will give some further arguments indicating that the choice $\beta = -1$ is the physically correct one. So, for the time being, we will set $\alpha = 0$ and $\beta = -1$ and discuss a few issues related to the exchange symmetry $4\pi S \leftrightarrow T$.

The non-perturbative exchange symmetry $4\pi S \leftrightarrow T$ is true for arbitrary U in the chamber $S, T > U$. As already discussed in detail in [12], at the fixed point $t_2 = S_{inv}^\infty = 0$ of this transformation, one has that $S = T > U$, the complex structure field y takes the value $y = 1$, and the discriminant of the Calabi-Yau model vanishes. The locus $S = T > U$ corresponds to a strong coupling singularity with additional massless states. In the model based on the Calabi-Yau space $WP_{1,1,2,8,12}(24)$, a non-Abelian gauge symmetry enhancement with an equal number of massless vector and hypermultiplets takes place at $S = T > U$, such that the non-Abelian β -function vanishes [28, 29].

On the other hand, the non perturbative exchange symmetry $4\pi S \leftrightarrow T$ implies that for $T \rightarrow \infty$ there is a ‘perturbative’ $4\pi S \leftrightarrow U$ exchange symmetry. This symmetry is nothing but the $T-S$ transformed perturbative symmetry (6). Furthermore, for $T \rightarrow \infty$, there is a modular symmetry $SL(2, \mathbf{Z})_S \times SL(2, \mathbf{Z})_U$ and the corresponding ‘perturbative’ monodromy matrices of the prepotential can be computed in a straightforward way. Hence, for $T \rightarrow \infty$, there is a ‘perturbative’ gauge symmetry enhancement of either $U(1)^2$ to $SU(2) \times U(1)$ or to $SU(2)^2$ or to $SU(3)$ at the points $S = U$, $S = U = 1$ or $S = U = e^{i\pi/6}$, respectively, with no additional massless hyper multiplets [15].

Let us now investigate the gravitational coupling F_1 , again first in the context of type II compactifications. It can be expressed in terms of the Kähler moduli fields t_i as the following instanton sum [30]

$$F_1^{II} = -i \sum_{i=1}^h t_i c_2 \cdot J_i - \frac{1}{\pi} \sum_n \left[12 n_{d_1, \dots, d_h}^e \log(\tilde{\eta}(\prod_{i=1}^h q_i^{d_i})) + n_{d_1, \dots, d_h}^r \log(1 - \prod_{i=1}^h q_i^{d_i}) \right]. \quad (22)$$

Here $\tilde{\eta}(q) = \prod_{m=1}^\infty (1 - q^m)$, and the n_{d_1, \dots, d_h}^e denote the elliptic genus one instanton numbers. We will again specialize to the Calabi-Yau space $WP_{1,1,2,8,12}(24)$ with $h = 3$. Then the non-exponential piece, which dominates for large t_i , reads [28]

$$-i \sum_{i=1}^3 t_i c_2 \cdot J_i = 92t_1 + 24t_2 + 48t_3. \quad (23)$$

This expression is explicitly invariant under the non-perturbative symmetry (7). Furthermore, by also explicitly checking some of the elliptic instanton numbers n_{d_1, d_2, d_3}^e , one discovers that, just like in the case of n^r ,

$$n_{d_1, 0, d_3}^e = n_{d_1, 0, d_1 - d_3}^e \quad \text{and} \quad n_{d_1, d_2, d_3}^e = n_{d_1, d_3 - d_2, d_3}^e. \quad (24)$$

It follows that F_1^{II} is symmetric under the two exchange symmetries (6) and (7).

In the heterotic case the holomorphic gravitational coupling at the one-loop level is given by

$$F_1^{het} = 24S_{inv} + \frac{b_{grav}}{8\pi^2} \log \eta^{-2}(T)\eta^{-2}(U) + \frac{2}{4\pi^2} \log(j(T) - j(U)). \quad (25)$$

For the model we are discussing one has that $b_{grav} = 48 - \chi = 528$. Inserting S_{inv} given in (13) into F_1^{het} yields [15]

$$\begin{aligned} F_1^{het} &= 24 \left(S - \frac{1}{768\pi^2} \partial_T \partial_U \left(\tilde{d}_{abc}^{2,2} y^a y^b y^c \right) - \frac{1}{8\pi^2} \log(j(T) - j(U)) \right. \\ &\quad \left. + \frac{1}{8\pi^2} \sum_{k,l \geq 0} k l c_1(kl) L_{i_1}(e^{-2\pi(kT+lU)}) - \frac{1}{8\pi^2} L_{i_1}(e^{-2\pi(T-U)}) \right) \\ &\quad + \frac{b_{grav}}{8\pi^2} \log \eta^{-2}(T)\eta^{-2}(U) + \frac{2}{4\pi^2} \log(j(T) - j(U)). \end{aligned} \quad (26)$$

Let us now compare the heterotic and the type II gravitational couplings.⁴ We will match $4\pi F_1^{het}$ with F_1^{II} . First take the decompactification limit to $D = 5$, i.e. the limit $T, U \rightarrow \infty$ ($T > U$). This eliminates all instanton contributions, i.e. all exponential terms. In the heterotic case we get in this limit

$$F_1^{het} \rightarrow F_1^\infty = 24S_{inv}^\infty + \frac{12}{\pi} T + \frac{11}{\pi} U = 24S + \frac{12 + 6\beta}{\pi} T + \frac{11 + 6\alpha}{4\pi} U. \quad (27)$$

By comparing this expression with the type II large t_i limit given in (23), one finds that (23) and (27) match up precisely for the identification given in (16) between heterotic and type II moduli. When choosing $\alpha = 0$ and $\beta = -1$, it follows that F_1^∞ is symmetric under the exchange $4\pi S \leftrightarrow T$.⁵ This symmetry implies that in the limit $T \rightarrow \infty$, F_1^{het} can be written [15] in terms of $SL(2, \mathbf{Z})_S$ modular functions $j(4\pi S)$ and $\eta(4\pi S)$ by just replacing $4\pi S$ with T in equation (26).

Next, let us compare the exponential terms in F_1^{II} and F_1^{het} . In the type II case we have to consider the weak coupling limit $q_2 \rightarrow 0$; hence only the terms with the instanton numbers $n_{d_1,0,d_3}^{r,e}$ contribute to the sum. We will see that, when comparing with the heterotic expression, one gets a very interesting relation between the rational and elliptic instanton numbers for $d_2 = 0$. In order to do this comparison, we have to recall that $Li_1(e^{-2\pi(kT+lU)}) = -\log(1 - e^{-2\pi(kT+lU)})$. The difference $j(T) - j(U)$ can be written in

⁴It was already, to some extent, shown in [16, 18, 31] that the heterotic and type gravitational couplings agree.

⁵In [15] a different choice was made for these two parameters, namely $\alpha = -11/6$ and $\beta = -2$. Hence it follows that $F_1^\infty = 24S$.

the following useful form (in the chamber $T > U$) [32, 19]

$$\log(j(T) - j(U)) = 2\pi T + \sum_{k,l} c(kl) \log(1 - e^{-2\pi(kT+lU)}), \quad (28)$$

where the integers k and l can take the following values [19]: either $k = 1, l = -1$ or $k > 0, l = 0$ or $k = 0, l > 0$ or $k > 0, l > 0$. The universal constants $c(n)$ are defines as follows:

$$\begin{aligned} j(q) - 744 &= \sum_{n=-1}^{\infty} c(n)q^n = \frac{1}{q} + 196884q + 21493760q^2 + 864299970q^3 \\ &+ 20245856256q^4 + \dots \end{aligned} \quad (29)$$

First consider the terms with $k = 1, l = -1$ on the heterotic side. Matching the term $\log(1 - e^{-2\pi(T-U)})$ contained in $4\pi F_1^{het}$ with F_1^{II} requires that

$$10c(-1) - 12c_1(-1) = 12n_{0,0,1}^e + n_{0,0,1}^r. \quad (30)$$

This is indeed satisfied, since $c(-1) = c_1(-1) = 1$ and $n_{0,0,1}^e = 0, n_{0,0,1}^r = -2$.

Next, consider the terms in the sum with $k > 0, l = 0$ (and analogously $k = 0, l > 0$). Since $c(0) = 0$, only the term $\frac{b_{grav}}{8\pi^2} \log \eta^{-2}(T)$ contributes on the heterotic side ($b_{grav} = 528$). Matching $4\pi F_1^{het}$ with F_1^{II} yields the following relation among the instanton numbers ($d_1 = d_3 = k$):

$$12 \sum_{i=1}^s n_{k_i,0,k_i}^e + n_{k,0,k}^r = b_{grav} = 528. \quad (31)$$

The k_i ($i = 1, \dots, s$) are the divisors of k ($k_1 = k, k_s = 1$). Using Klemm's list of explicit instanton numbers, we checked that this relation is indeed true up to $k = 4$ ($n_{1,0,1}^e = 4, n_{k,0,k}^e = 0$ for $k > 1, n_{k,0,k}^r = -\chi = 480$).

Finally, consider the case where $k > 0, l > 0$. By comparing the heterotic and type II expressions we derive the following interesting relation ($d_1 = k + l, d_3 = k$):

$$\begin{aligned} 12 \sum_{i=1}^s n_{d_1^i,0,d_3^i}^e &= -n_{k+l,0,k}^r + 10c(kl) + 12klc_1(kl) = \\ &= 10c(kl) + (12kl + 2)c_1(kl). \end{aligned} \quad (32)$$

Here s is the number of common divisors m_i ($i = 1, \dots, s$) of $d_1 = k + l$ and $d_3 = k$ with $d_1^i = d_1/m_i$ and $d_3^i = d_3/m_i$ (where $m_1 = 1$). Again we can explicitly check the non-trivial relation (32) for the first few terms. For example, for $k = l = 1$ one has

$$n_{2,0,1}^e = -948 \quad \text{and} \quad n_{2,0,1}^r = 282888, \quad (33)$$

which, together with equations (11) and (29), confirms the above relation. For $k = 2$ and $l = 1$ one finds that $12n_{3,0,2}^e + n_{3,0,2}^r = 10c(2) + 24c_1(2)$ is indeed satisfied, since

$$n_{3,0,2}^e = -568640 \quad \text{and} \quad n_{3,0,2}^r = 17058560. \quad (34)$$

And finally, for $k = l = 2$ for instance, one finds that the relation $12(n_{4,0,2}^e + n_{2,0,1}^e) + n_{4,0,2}^r = 10c(4) + 48c_1(4)$ indeed holds due to

$$n_{4,0,2}^e = -1059653772 \quad , \quad n_{2,0,1}^e = -948 \quad \text{and} \quad n_{4,0,2}^r = 8606976768. \quad (35)$$

Now consider the relation $\left(\frac{E_6 E_4}{\eta^{24}}\right)' = -\frac{2\pi}{6}\left(\frac{E_2 E_4 E_6}{\eta^{24}} + 2\frac{E_6^2}{\eta^{24}} + 3\frac{E_4^3}{\eta^{24}}\right)$. From this we can, for $n > 0$, derive the useful equation $12nc_1(n) + 10c(n) = -2\tilde{c}_1(n)$, where the $\tilde{c}_1(n)$ are defined as follows

$$E_2 \frac{E_6 E_4}{\eta^{24}}(q) = \sum_{n=-1}^{\infty} \tilde{c}_1(n) q^n = \frac{1}{q} - 264 - 135756q - 5117440q^2 + \dots \quad (36)$$

It follows that one can rewrite equation (32) as

$$12 \sum_{i=1}^s n_{d_1^i, 0, d_3^i}^e + n_{d_1, 0, d_3}^r = 10c(kl) + 12klc_1(kl) = -2\tilde{c}_1(kl) \quad , \quad k > 0, l > 0. \quad (37)$$

This corresponds to the following integral⁶ of [19] (with $b_{grav} = 48 - \chi = -2(c_1(0) - 24) = -2\tilde{c}_1(0)$)

$$\begin{aligned} \tilde{I}_{2,2} &= \frac{-1}{2} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \left[-\frac{i}{\eta^2} T r_R J_0 (-1)^{J_0} q^{L_0 - 22/24} \bar{q}^{\bar{L}_0 - 9/24} \left(E_2 - \frac{3}{\pi\tau_2} \right) - b_{grav} \right] \\ &= \frac{-1}{2} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \left[-2Z_{2,2} \frac{E_4 E_6}{\eta^{24}} \left(E_2 - \frac{3}{\pi\tau_2} \right) - (-2\tilde{c}_1(0)) \right] \end{aligned} \quad (38)$$

Let us briefly summarize our results obtained so far. The type II prepotential F_0^{II} is determined by rational (genus 0) instanton numbers n^r . Comparison with the semiclassical heterotic prepotential F_0^{het} relates a subset of the rational instanton numbers n^r ($d_2 = 0$) to the coefficients of the modular function $\frac{E_4 E_6}{\eta^{24}}$ of modular weight -2. The type II gravitational coupling F_1^{II} depends, in addition, on the elliptic (genus 1) instanton numbers n^e . A subset of those can be expressed in terms of the coefficients of the modular functions $\frac{E_2 E_4 E_6}{\eta^{24}}$ and $\frac{E_4 E_6}{\eta^{24}}$. For the higher F_g (cf. [33]) we conjecture the following. Under modular transformations $T \rightarrow \frac{1}{T}$ the F_g transform at weak coupling as

$$F_g \rightarrow T^{2(g-1)} F_g, \quad (39)$$

⁶The precise relation of this integral to F_1^{het} was worked out in [15].

i.e. F_g has modular weight $2(g-1)$. Thus we are tempted to conclude that the higher (genus g) instanton numbers are determined by the coefficients of a modular form of modular weight $2(g-1)$. Since the ring of modular functions, together with η^{-24} is finite, only a finite number of different types of instanton numbers seem to be independent.

Note that such a fact is known for the case of a one dimensional Calabi-Yau target space, that is an elliptic curve, where according to (also cf. [34, 35, 36]) the F_g are quasimodular forms of weight $6g-6$ for $g \geq 2$, i.e. $F_g \in \mathbf{Q}[E_2, E_4, E_6]$. Also note that one has $F_1 = -\log \eta$ [30]. This is here conjecturally extended to the elliptically fibered Calabi-Yau space considered above.

3. Comments on other Calabi-Yau models

At the end, let us briefly consider different Calabi-Yau spaces and also comment on the relation to the heterotic/heterotic duality in six dimensions [7, 8], with the 6-dimensional heterotic string compactified on K_3 . (The decompactification limit from $D=4$ to $D=6$ is obtained by sending $T \rightarrow \infty$ with U finite; as discussed in [20, 7], the $D=6$ heterotic/heterotic duality becomes an exchange symmetry of S with T in $D=4$.) We will concentrate on three families of CY's with Hodge numbers $(3,243)$, which are elliptic fibrations over \mathbf{F}_n with $n=0,1,2$ [11, 37]. Being elliptic fibrations, they can be used to compactify F -theory to six dimensions. In the $D=6$ heterotic string, the integer n is related to the number s of $SU(2)$ instantons in one of the two E_8 's by $n=s-12$ [11, 37]. First consider the case of an elliptic fibration over \mathbf{F}_0 , corresponding to the symmetric embedding of the $SU(2)$ bundles with equal instanton numbers $s=s'=12$ into $E_8 \times E'_8$. This leads to a $D=6$ heterotic model with gauge group $E_7 \times E'_7$ with $\tilde{v}_\alpha = \tilde{v}'_\alpha = 0$ [7]. There are 510 hypermultiplets transforming as $4(56,1) + 4(1,56) + 62(1,1)$. The heterotic/heterotic duality originates from the existence of small instanton configurations [38]. The model is, however, not self-dual, since the non-perturbative gauge groups appear in different points of the hyper multiplet moduli space than the original gauge groups [7]. For generic vev's of the hyper multiplets the gauge group is completely broken and one is left with 244 hyper multiplets and no vector multiplets. Upon compactification to $D=4$ on T_2 one arrives at the heterotic string with gauge group $U(1)^4$, which is the dual to the considered type II string on the CY $WP_{1,1,2,8,12}(24)$. Semiclassically, at special points in the hypermultiplet moduli space, this gauge group can be enhanced to a non-Abelian gauge group, inherited from the $E_7 \times E'_7$ with $N=2$ β -function coefficient $b_\alpha = 12(1 + \frac{\tilde{v}_\alpha}{v_\alpha}) = 12$ [8].

Next, consider embedding the $SU(2)$ bundles in an asymmetric way into the two E_8 's [39, 8]: $s = 14$, $s' = 10$. This corresponds to the elliptic fibration over \mathbf{F}_2 [11, 37]. Note that \mathbf{F}_0 and \mathbf{F}_2 are of the same parity (even N), so they are connected by deformation [11], as we will discuss in the following. Then, in this case, one has [8] a gauge group $E_7 \times E'_7$ with $\tilde{v}_\alpha = 1/6$ and $\tilde{v}'_\alpha = -1/6$ and hyper multiplets transforming as $5(56, 1) + 3(1, 56) + 62(1, 1)$. The second E_7 can be completely Higgsed away, leading to a $D = 6$ heterotic model with gauge group E_7 and hypermultiplets $5(56) + 97(1)$. As explained in [8], this model also possesses a heterotic/heterotic duality, however without involving non-perturbative small instanton configurations. Hence in this sense, this model is really self-dual. Just like in the case of the symmetric embedding, the gauge group E_7 is spontaneously broken for arbitrary vev's of the gauge non-singlet hyper multiplets and one is again left with 244 hyper multiplets and no vector multiplet. When compactifying on T_2 to $D = 4$, one obtains the same heterotic string model with $U(1)^4$ gauge group as before. For special values of the hyper multiplets a non-Abelian gauge group is obtained, now however with β -function coefficient $b_\alpha = 12(1 + \frac{\tilde{v}_\alpha}{v_\alpha}) = 24$ [8].

In summary, the symmetric (12,12) model and the asymmetric (14,10) model should be considered as being the same [8, 11], since both are related by the Higgsing and both lead to the same heterotic string in $D = 4$.

As already mentioned, we would like to provide a six-dimensional argument for why $\beta = -1$ is the physically correct choice for one of the cubic parameters. We will directly follow the discussion given in [8] and consider the one-loop gauge coupling for the enhanced non-Abelian gauge groups that are inherited from the six-dimensional gauge symmetries. Specifically, the gauge kinetic function is of the form [24, 8]

$$f_\alpha = S_{inv} - \frac{b_\alpha}{8\pi^2} \log(\eta(T)\eta(U))^2. \quad (40)$$

Using equation (14) this then becomes in the decompactification limit $T \rightarrow \infty$ to $D = 6$

$$f_\alpha \rightarrow S + \frac{1 + \beta + \frac{\tilde{v}_\alpha}{v_\alpha}}{4\pi} T. \quad (41)$$

By comparing this expression with the six-dimensional gauge coupling [40], it then follows that $\beta = -1$.

Let us also make some remarks on the third model with Hodge numbers (3,243), the (13,11) embedding [11, 37]. This is now elliptically fibered over \mathbf{F}_1 . In going to the Higgs branch [37] one reaches the Calabi-Yau $WP_{1,1,1,6,9}(18)$ with $h^{1,1} = 2$, $h^{2,1} = 272$. Note that in the $D = 6$ interpretation of F-theory on this Calabi-Yau, this corresponds to losing a tensor multiplet and gaining 29 hyper multiplets. Thus, in four dimensions,

the two vector multiplets correspond to T and U . No dilaton S is present, reflecting the fact that this CY is not a K_3 fibration, and no heterotic dual (at weak coupling) exists. This CY is now elliptically fibered over \mathbf{P}^2 (the exceptional curve of \mathbf{F}_1 was blown down) [37]. According to [41], the rational instanton numbers $n_{j,0}^r$ of this CY are all equal to $540 = -\chi_{CY}$. Thus, compared to the CY $WP_{1,1,2,8,12}(24)$, the corresponding modular form is now simply a constant. For $q_2 = 0$, the Yukawa coupling y_{111} of the $WP_{1,1,1,6,9}(18)$ model is given by E_4 [41]. This can be nicely compared with the following Yukawa coupling [24] of the CY $WP_{1,1,2,8,12}(24)$ model in the limit $T \rightarrow \infty$

$$\partial_U^3 h^{(1)} \sim \frac{E_4(U)}{j(T) - j(U)} \frac{E_4(T)E_6(T)}{\eta^{24}(T)} \rightarrow E_4(U). \quad (42)$$

The elliptic instanton numbers in the $WP_{1,1,1,6,9}(18)$ model satisfy the following relation: $12 \sum n_{j,0}^e + n_{j,0}^r = 12 \cdot 3 + 540$. Here, $n_{j,0}^e = 3$ (versus $n_{j,0}^e = 4$ in the $WP_{1,1,2,8,12}(24)$ model) is determined by the elliptic fibration base with $\chi(\mathbf{P}^2) = 3$ (versus $\chi(\mathbf{F}_1) = 4$ in the $WP_{1,1,2,8,12}(24)$ model). This difference in the $n_{j,0}^e$ corresponds to the loss of one $h^{1,1}$ class (of the elliptic fibration base or equally well of the whole Calabi-Yau) in the blowing down process. Accordingly, the expression $b_{grav} = 48 - \chi(CY)$ is modified.

It is instructive to consider the mirror map of this model. Using the complex structure variable $Y_1 = -\frac{1}{X_1}$ (cf. chapter 7.2 in [41]) the mirror map becomes $\frac{1}{Y_1(1-432Y_1)} = j(U)$ for $T \rightarrow \infty$, which corresponds to the elliptic family $P_{1,2,3}(6)$ in [12]. This also corresponds to the $T \rightarrow \infty$ limit (at $S \rightarrow \infty$) of the $S - T - U$ Calabi-Yau model $WP_{1,1,2,8,12}(24)$, which can also be considered to be elliptically fibered by the elliptic family $P_{1,2,3}(6)$ over \mathbf{F}_0 , when one considers [42, 11] the one nonpolynomial deformation of $WP_{1,1,2,8,12}(24)$, which deforms the base \mathbf{F}_2 to \mathbf{F}_0 .

Let us close with the following remark. The $S - T$ exchange symmetry is also present [12] in the $S - T$ model based on the CY $WP_{1,1,2,2,6}(12)$ with Hodge numbers $h_{1,1} = 2, h_{2,1} = 128$. This model, however, falls out of the class of the CY's considered above, since, even though being a K_3 -fibration, it does not correspond to an elliptic fibration. This model is obtained [6] by first performing a toroidal compactification to $D = 8$ on a torus with $T = U$ and enhanced $SU(2)$ gauge group, and subsequently going down to $D = 4$ by a K_3 compactification. Like in the case of [7], there is again a symmetric embedding of the $SU(2)$ gauge bundle into $E_8 \times E'_8 \times SU(2)$: $(s, s', s'') = (10, 10, 4)$. The $S - T$ exchange symmetry, however, is not related to a six-dimensional heterotic/heterotic duality or to F -theory on a CY.

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